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On a representable class of separated lattice EQ-algebras

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Abstract

In this paper, we introduce and study a class of separated lattice EQ-algebras that may be represented as subalgebras of products of linearly ordered ones. Such algebras are called representable. Namely, we enrich separated lattice EQ-algebras with a unary operation (the so called Baaz delta), fulfilling some additional assumptions. The resulting algebras are called $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. One of the main results of this paper is to characterize the class of representable $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. We also supply a number of useful results, leading to this characterization.

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1. Introduction

EQ-algebras were introduced by Novák (2006) [1] and Novák and Baets (2009) [2] as generalization of residuated lattices (see [3]). Unlike the residuated lattices, the basic operation in it is a fuzzy equality while implication is derived from it. Its original motivation comes from the study of higher-order fuzzy logic [4] that was obtained as a generalization of simple type theory in the style of L. Henkin who developed in [5] a very elegant theory (cf. also [6]) in which the basic connective is equality.

EQ-algebras brought an idea to develop (fuzzy) many-valued logics on the basis of fuzzy equality (equivalence) as the principal connective. Accordingly, a formal theory of new different many-valued logics, called EQ-logics, has been recently introduced by M. Dyba and V. Novák [7].

The current investigation of EQ-algebras (see [7-10]) shows that goodness, i.e. each element x is equal to $\mathbf{1}$ in the degree x, is sufficient for the resulting algebra has many reasonable properties. The goodness axiom implies that the algebra is separated (i.e., two elements equal in the degree $\mathbf{1}$ must be identical) but not vice-versa. Therefore, Separateness turned out to be indispensable for any kind of fuzzy equality-based logic.

One of the important algebraic consequences of goodness axiom is axiomatizing the class of representable good EQ-algebras (expanded by Delta-connective) (see [8, 9]). This is mainly based on the fact that good EQ-algebras give raise to BCK-algebras [11, 12].

In this paper, we continue the study of EQ-algebras. We introduce and study a class of separated (not necessarily good) lattice EQ-algebras that may be represented as subalgebras of products of linearly ordered ones. Such algebras are called representable. Namely, we enrich separated lattice EQ-algebras with a unary operation (the so called Baaz delta), fulfilling some additional assumptions. The resulting algebras are called $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. One of the main results of this paper is to characterize the class of representable $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. We show that prelinearity alone characterizes the representable class of $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras. We also supply a number of useful results, leading to this characterization.

This paper is structured as follows: in the next section we overview the basic definitions and properties of EQ-algebras and their special. In Section 4 we introduce and study the class of $\ell\mathrm{EQ}_{\Delta}^{\mathrm{s}}$-algebras and we devote Section 5 to summarize the results.

2. EQ-algebras: an overview

Definition 1. ([9]) An algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, \mathbf{1})$ of type (2, 2, 2,0) is called an EQ-algebra where for all $a, b, c, d \in E$:

(E1) $(E, \wedge, \mathbf{1})$ is a $\wedge$-semilattice with top element $\mathbf{1}$. We set $a \leq b$ iff $a \wedge b = a$,
(E2) $(E, \otimes, \mathbf{1})$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \leq b$,
(E3) $a \sim a = \mathbf{1}$, (reflexivity)
(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$, (substitution)
(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$, (congruence)
(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$, (monotonicity)
(E7) $a \otimes b \leq a \sim b$.

The binary operation " $\wedge$ " is called meet (infimum), " $\otimes$ " is called multiplication, and " $\sim$ " is a fuzzy equality. We set, for $a, b \in E$:

$$a \to b = (a \wedge b) \sim a \qquad (1)$$
$$\tilde{a} = a \sim \mathbf{1} \qquad (2)$$

The derived operation (1) will be called *implication*. If $\mathbf{0}$ is a bottom element of E, then we define the unary operation $\neg$ on E, for all $a \in E$, by

$$\neg a = a \sim \mathbf{0} \qquad (3)$$

Definition 2. ([4, 8]) Let $\mathcal{E}$ be an EQ-algebra. We say that it is:

(a) *separated* if for all $a, b \in E, a \sim b = \mathbf{1}$ implies $a = b$,

(b) *good* if for all $a \in E$, $\tilde{a} = a$,

(c) *lattice* EQ-algebra (ℓEQ-algebra) if the underlying $\wedge$-semilattice is a lattice in which the following substitution axiom holds for all $a, b, c, d \in E$:

$$((a \vee b) \sim c) \otimes (d \sim a) \leq (d \vee b) \sim c \qquad (4)$$

(d) *prelinear* if for all $a, b \in E$, $\mathbf{1}$ is the unique upper bound in E of the set $\{(a \to b), (b \to a)\}$.

Note that every good EQ-algebra is separated, but not vice-versa (see [4]). EQ-algebra has many interesting properties (see [4, 9]). We only mention some of them that will used later.

Lemma 1. ([9, 13]) Let $\mathcal{E}$ be an EQ-algebra. Then the following properties hold for all $a, b, c \in E$:

(a) $a \sim b = b \sim a$;

(b) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$;

(c) $b \leq \tilde{b} \leq a \to b$;

(d) $a \otimes b \leq a \wedge b \leq a, b$;

(e) $(a \sim b) \leq a \to b$ *and* $a \to a = \mathbf{1}$;

(f) If $a \leq b$ then $a \to b = \mathbf{1}$, $c \to a \leq c \to b$ and $b \to c \leq a \to c$;

(g) $(a \to b) \leq (c \to a) \to (c \to b)$;

(h) $(a \to b) \leq (b \to c) \to (a \to c)$;

(i) $a \to (b \to c) \leq b \to (a \to \tilde{c})$;

(j) $a \to (b \to c) \leq (a \otimes b) \to \tilde{c}^4$;

(k) If $\mathcal{E}$ is ℓEQ-algebra, then $(a \to c) \otimes (b \to c) \leq (a \vee b) \to c$.

Proposition 1. ([9]) The following statements are equivalent:

(a) An EQ-algebra $\mathcal{E}$ is separated.

(b) $a \leq b$ iff $a \to b = \mathbf{1}$ for all $a, b \in E$.

This means that the implication operation " $\to$ " in a separated EQ-algebra precisely reflects the ordering " $\leq$ ".

3. ℓEQ$^s_\Delta$-algebras

Definition 3. A ℓEQ$^s_\Delta$-algebra is an algebra $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ that is a separated ℓEQ-algebra with a bottom element $\mathbf{0}$ expanded by a unary operation $\Delta: E \to E$ fulfilling the following axioms:

(EΔ1) $\Delta\mathbf{1} = \mathbf{1}$;

(EΔ2) $\Delta a \leq a$;

(EΔ3) $\Delta a \leq \Delta\Delta a$;

(EΔ4) $\Delta(a \sim b) \leq \Delta a \sim \Delta b$;

(EΔ5) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$;

(EΔ6) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$;

(EΔ7) $\Delta a \vee \neg\Delta a = \mathbf{1}$;

(EΔ8) $\Delta(a \sim b) \leq (a \otimes c) \sim (a \otimes c)$;

(EΔ9) $\Delta(a \sim b) \leq (c \otimes a) \sim (c \otimes b)$.

Note that the axioms (EΔ1), (EΔ2),…, (EΔ7) are from [9], and the two inequalities (EΔ8) and (EΔ9) are from [10]. They are necessary to assure good behavior of the multiplication with respect to the crisp equality. If we omit "Δ" in (EΔ8) and (EΔ9) then the resulting EQ-algebra becomes residuated (see [9]).

Lemma 2. Let $\mathcal{E}_\Delta$ be a ℓEQ$^s_\Delta$-algebra. For all $a, b, c \in E$, it holds that:

(a) If $a \leq b$, then $\Delta a \leq \Delta b$;

(b) $\Delta(a \to b) \leq \Delta a \to \Delta b$;

(c) $\Delta(a \vee b) = \Delta a \vee \Delta b$;

(d) $\Delta\Delta a = \Delta a$;

(e) $a \otimes \Delta(a \to b) \leq b$, $\Delta(a \to b) \otimes a \leq b$;

(f) $a \otimes \Delta(a \sim b) \leq b$, $\Delta(a \sim b) \otimes a \leq b$;

(g) $\Delta(a \sim \mathbf{1}) = \Delta a$ and $\Delta(\mathbf{1} \to a) = \Delta a$;

(h) $\Delta b \leq c \to (b \otimes c)$ and $\Delta b \leq c \to (c \otimes b)$;

(i) $\Delta a = \Delta a \otimes \Delta a$;

(j) $\Delta a \leq \Delta b \to \Delta c$ iff $\Delta a \otimes \Delta b \leq \Delta c$ and $\Delta b \otimes \Delta a \leq \Delta c$;

(k) If $\mathcal{E}_\Delta$ is prelinear, then $\Delta(a \to b) \vee \Delta(b \to a) = \mathbf{1}$;

(l) $\Delta(a \to b) \leq (a \otimes c) \to (b \otimes c)$, and $\Delta(a \to b) \leq (c \otimes a) \to (c \otimes b)$.

Proof. (a): Assume $a \leq b$ $(a \wedge b = a)$. Hence, by (EΔ5), we have

$\Delta(a \wedge b) = \Delta a \wedge \Delta b = \Delta a$; that is $\Delta a \leq \Delta b$.

(b): From (EΔ4) and (EΔ5), we get

$$\Delta(a \to b) = \Delta((a \wedge b) \sim a) \leq \Delta(a \wedge b) \sim \Delta b = (\Delta a \wedge \Delta b) \sim \Delta b = \Delta a \to \Delta b.$$

(c): From item (a) (because $a, b \leq a \vee b$), we can have, $\Delta a, \Delta b \leq \Delta(a \vee b)$. Therefore, $\Delta a \vee \Delta b \leq \Delta(a \vee b)$. Hence, by this and (EΔ6), the result holds.

(d): Direct from (EΔ2) with item (a), we obtain $\Delta\Delta a \leq \Delta a$. Hence, by this and (EΔ3), the result holds.

(e): From (EΔ2), Lemma 1(d) and the order properties of " → ", we get

$$\Delta(a \to b) \leq (a \to b) \leq (a \otimes \Delta(a \to b)) \to b,$$
$$\neg\Delta(a \to b) = \Delta(a \to b) \to \mathbf{0} \leq \Delta(a \to b) \to b \leq (a \otimes \Delta(a \to b)) \to b$$

(since $\mathbf{0} \leq b$). Thus, by (EΔ7) and Proposition 1,

$(a \otimes \Delta(a \to b)) \to b = \mathbf{1}$; that is $(a \otimes \Delta(a \to b)) \leq b$.

Similarly, $\Delta(a \to b) \otimes a \leq b$.

(f): Directly from item (e) by Lemma 1(e).

(g): By item (d), (EΔ4) and item (f), we get

$$\Delta(a \sim \mathbf{1}) = \Delta\Delta(a \sim \mathbf{1}) = \mathbf{1} \otimes \Delta\Delta(\mathbf{1} \sim a) \leq \Delta\mathbf{1} \otimes \Delta(\Delta\mathbf{1} \sim \Delta a) \leq \Delta a.$$

On the other hand, $\Delta a \leq \Delta(a \sim \mathbf{1})$ by item (a) (since $a \leq (a \sim \mathbf{1})$).

In particular, $\Delta(\mathbf{1} \to a) = \Delta((\mathbf{1} \wedge a) \sim \mathbf{1}) = \Delta(a \sim \mathbf{1}) = \Delta a$.

(h): From item (g), (EΔ8) and Lemma 1(e), we get

$$\Delta b = \Delta(\mathbf{1} \sim b) \leq (\mathbf{1} \otimes c) \sim (b \otimes c) \leq (\mathbf{1} \otimes c) \to (b \otimes c) = c \to (b \otimes c).$$

Similarly, $\Delta b \leq c \to (c \otimes b)$.

(i): By item (h), item (d) and order properties of " → ", we obtain

$\Delta a = \Delta\Delta a \leq \Delta a \to (\Delta a \otimes \Delta a)$ and

$\neg\Delta a = \Delta a \to \mathbf{0} \leq \Delta a \to (\Delta a \otimes \Delta a)$

(since $\mathbf{0} \leq (\Delta a \otimes \Delta a)$). Thus, by (EΔ7) and Proposition 1, $\Delta a \to (\Delta a \otimes \Delta a) = \mathbf{1}$; that is $\Delta a \leq (\Delta a \otimes \Delta a)$. On the other hand, $(\Delta a \otimes \Delta a) \leq \Delta a$ by Lemma 1(d).

(j): Assume $\Delta a \leq \Delta b \to \Delta c$, then by Lemma 1(d) and the order properties of " → ",

$\Delta a \leq \Delta b \to \Delta c \leq (\Delta a \otimes \Delta b) \to \Delta c$ and

$\neg\Delta a = \Delta a \to \mathbf{0} \leq \Delta a \to \Delta c \leq (\Delta a \otimes \Delta b) \to \Delta c$.

Thus, by (EΔ7), and Proposition 1, $(\Delta a \otimes \Delta b) \to \Delta c = \mathbf{1}$; that is $(\Delta a \otimes \Delta b) \leq \Delta c$. Similarly, $(\Delta b \otimes \Delta a) \leq \Delta c$. Conversely, assume $(\Delta a \otimes \Delta b) \leq \Delta c$. Hence, by item (d) and item (h), we obtain

$$\Delta a = \Delta\Delta a \leq \Delta b \to (\Delta a \otimes \Delta b) \leq \Delta a \to \Delta c.$$

Similarly, for $(\Delta b \otimes \Delta a) \leq \Delta c$.

(k): By (EΔ1), the prelinearity and item (c), we get

$$\mathbf{1} = \Delta\mathbf{1} = \Delta((a \to b) \vee (b \to a)) = \Delta(a \to b) \vee \Delta(b \to a).$$

(l): Using (EΔ8) and the order properties of " → ", we have

$$\Delta(a \to b) = \Delta((a \wedge b) \sim a) \leq ((a \wedge b) \otimes c) \sim (a \otimes c) \leq (a \otimes c) \to ((a \wedge b) \otimes c) \leq (a \otimes c) \to (b \otimes c).$$

Similarly, $\Delta(a \to b) \leq (c \otimes a) \to (c \otimes b)$. ∎

Definition 4. Let $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ be a $\ell\mathrm{EQ}_\Delta^s$-algebra. A subset $F \subseteq E$ is called a *filter* of $\mathcal{E}_\Delta$ if for all $a, b \in E$:

(a) $\mathbf{1} \in F$.

(b) if $a, a \to b \in F$, then $b \in F$.

(c) if $a \in F$, then $\Delta a \in F$.

Note that a (prime) filer F on a $\ell\mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta = (E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1})$ is a (prime) prefilter (in the sense given in [9]) on its separated EQ-algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, \mathbf{1})$ satisfying (c). So all the properties of (prime) prefilters on a separated EQ-algebra (see [8, 9]) are also properties of (prime) filers on a $\ell\mathrm{EQ}_\Delta^s$-algebra, including the following result:

Lemma 3. (see [9]) Let F be a filter of a $\ell\mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$. For all $a, b \in E$ it holds that:

(a) If $a \in F$ and $a \leq b$ then $b \in F$;

(b) If $a, a \sim b \in F$ then $b \in F$;

(c) If $a, b \in F$ then $a \wedge b \in F$.

Lemma 4. Let F be a filter of a $\ell\mathrm{EQ}_\Delta^s$-algebra $\mathcal{E}_\Delta$. For all $a, b, c, a', b' \in E$ such that $a \sim b \in F$ and $a' \sim b' \in F$, it holds that

(a) If $a \to b \in F$, then $(a \otimes c) \to (b \otimes c) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$

(b) If $a, b \in F$ then $a \otimes b \in F$;

(c) $(a \otimes a') \sim (b \otimes b') \in F$ and $(a' \otimes a) \sim (b' \otimes b) \in F$;

(d) $(\Delta a \sim \Delta b) \in F$.

Proof. (a): Assume $a \to b \in F$. Since F is a filter, then $\Delta(a \to b) \in F$. Hence, by Lemma 2(l) and Lemma 3(a), we get

$\Delta(a \to b) \leq (a \otimes c) \to (b \otimes c) \in F$.

Similarly, $(c \otimes a) \to (c \otimes b) \in F$.

(b): From Lemma 1(c) and Lemma 3(a), it follows that $b \leq \mathbf{1} \to b \in F$. From item (a), it then follows that

$(a \otimes \mathbf{1}) \to (a \otimes b) = a \to (a \otimes b) \in F$.

Hence, by Definition 4 of a filter, $a \otimes b \in F$.

(c): By Definition 4, $\Delta(a \sim b)$ and $\Delta(a' \sim b') \in F$. Thus, by (EΔ8) and (EΔ9), we get

$$\begin{aligned}\Delta(a \sim b) \otimes \Delta(a' \sim b') &\leq \\ &\leq ((a \otimes a') \sim (b \otimes a')) \otimes ((b \otimes a') \sim (b \otimes b')) \\ &\leq (a \otimes a') \sim (b \otimes b')\end{aligned}$$

Hence, by Lemma 3(a) and item (b), the result holds. Similarly, $(a' \otimes a) \sim (b' \otimes b) \in F$.

(d): By Definition 4 and Lemma 3(a)

$\Delta(a \sim b) \in F$ implies $\Delta a \sim \Delta b \in F$ (since $\Delta(a \sim b) \leq \Delta a \sim \Delta b$). ■

Lemma 5. Let $\mathcal{E}_\Delta$ be a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra. Given a filter $F \subseteq E$, the following relation on $\mathcal{E}_\Delta$ is a congruence relation:

$$a \approx_F b \text{ iff } a \sim b \in F \tag{5}$$

Proof. Indeed, axiom (E3), Lemma 1(a) and Lemma 1(b) guarantee that $\approx_F$ is an equivalence relation. As an immediate consequence of Lemma 4, all the operations of $\mathcal{E}_\Delta$ are compatible with the relation given by (5); that is

$a \approx_F b$ and $a' \approx_F b'$ imply $(a \wedge a') \approx_F (b \wedge b'), (a \vee b') \approx_F (b \vee b'), (a \sim a') \approx_F (b \sim b'), (a \otimes a') \approx_F (b \otimes b')$, and $(\Delta a \approx_F \Delta b)$.

Then, $\approx_F$ is a congruence relation. ■

Let $\mathcal{E}_\Delta$ be a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra. For $a \in E$, we denote its equivalence class with respect to $\approx_F$ by $[a]_F$ and by E/F the quotient set associated with $\approx_F$. Furthermore, we define the factor algebra

$$\mathcal{E}_\Delta/F = \langle E/F, \wedge_F, \vee_F, \otimes_F, \sim_F, \Delta_F, \mathbf{0}_F, \mathbf{1}_F \rangle.$$

in the standard way as follows:

$E/F = \{[a]_F \mid a \in E\}$, and the binary operations on E/F are defined by

$[a]_F \wedge_F [b]_F = [a \wedge b]_F$;
$[a]_F \vee_F [b]_F = [a \vee b]_F$;
$[a]_F \sim_F [b]_F = [a \sim b]_F$;
$[a]_F \otimes_F [b]_F = [a \otimes b]_F$;
$\Delta_F[a]_F = [\Delta a]_F$.

The top and the bottom elements are $\mathbf{1}_F = [\mathbf{1}]_F = \{b \in E \mid b \sim \mathbf{1} \in F\} = F$, $\mathbf{0}_F = [\mathbf{0}]_F = \mathbf{0}$, respectively.

Also, we can define a binary relation " $\leq_F$ " on E/F as follows:

$$[a]_F \leq_F [b]_F \text{ iff } [a]_F \wedge_F [b]_F = [a]_F \text{ iff } a \wedge b \approx_F a \text{ iff } a \to b \in F \tag{6}$$

Then, we have the following result. Its proof proceeds in a standard way.

Theorem 1. Let F be a filter of a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra $\mathcal{E}_\Delta$. The factor algebra $\mathcal{E}_\Delta/F = \langle E/F, \wedge_F, \vee_F, \otimes_F, \sim_F, \Delta_F, \mathbf{0}_F, \mathbf{1}_F \rangle$ is a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra, and the mapping $f: E \longrightarrow E/F$ defined by $f(a) = [a]_F$ is a homomorphism of $\mathcal{E}_\Delta$.

For a nonempty subset X of a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra $\mathcal{E}_\Delta$, the smallest filter of $\mathcal{E}_\Delta$ which contains X, i.e. $\bigcap\{F \in \mathcal{F}(\mathcal{E}_\Delta): X \subseteq F\}$ is said to be a filter of $\mathcal{E}_\Delta$ generated by X and will be denoted by $\langle X \rangle$. Obviously, if X is a filter then $\langle X \rangle = X$. It is clear that if $X_1 \subseteq X_2$, then $\langle X_1 \rangle \subseteq \langle X_2 \rangle$. If $X = Y \cup \{a\}$, we will write $\langle Y, a \rangle$ for $\langle X \rangle$. The set of non-negative integers will be denoted by ω, for $a, b \in E, n \in \omega$, we define $a \to^0 b = b, a \to^{n+1} b = a \to (a \to^n b)$. If $a = \mathbf{1}, a \to^{n+1} b$ is denoted by $\tilde{b}^{n+1}$.

The following theorem gives a characterization of a filter generated by a set.

Theorem 2. Let X be a nonempty subset of a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra $\mathcal{E}_\Delta$. Then

$$\langle X \rangle = \{a \in E: \Delta b_1 \to (\Delta b_2 \to \cdots (\Delta b_n \to a) \ldots)) = \mathbf{1}, \text{ for some } b_i \in X, n \in \omega\}.$$

Proof. Put $M = \{a \in E: \Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to a) \ldots)) = \mathbf{1}, \text{ for some } b_i \in X, n \in \omega\}$. Now, we show that M is a filter of $\mathcal{E}_\Delta$. Since all $b_i \in M, b_i \leq \mathbf{1}$, therefore by Lemma 2(a) and (EΔ1) $\Delta b_i \leq \Delta \mathbf{1} = \mathbf{1}$ so $\Delta b_i \to \mathbf{1} = \mathbf{1}$; i.e., $\mathbf{1} \in M$. Now, let $a, a \to b \in M$, then there exist $b_1, b_2, \ldots, b_n, b'_1, b'_2, \ldots, b'_m \in X$ such that

$\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to a) \ldots)) = \mathbf{1}$ and
$\Delta b'_1 \to (\Delta b'_2 \to \ldots (\Delta b'_m \to (a \to b)) \ldots)) = \mathbf{1}$

Hence, by Lemma 1(g), we have:

$$\begin{aligned}a \to b &\leq (\Delta b_n \to a) \to (\Delta b_n \to b) \\ &\leq (\Delta b_{n-1} \to (\Delta b_n \to a)) \to (\Delta b_{n-1} \to (\Delta b_n \to b)).\end{aligned}$$

By continuing this way, we get that

$$a \to b \leq (\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to a) \ldots)) \to (\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to b) \ldots)).$$

Then, by order properties of " $\to$ ", Lemma 2(a) and (EΔ1), we conclude that

$$\begin{aligned}a \to b &\leq \mathbf{1} \to (\Delta b_1 \to (\Delta b_2 \to \cdots (\Delta b_n \to b) \ldots)) \\ &\leq \Delta b_0 \to (\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to b) \ldots)),\end{aligned}$$

where $b_0 \in M$. Hence,

$$\Delta b'_m \to (a \to b) \leq \Delta b'_m \to \Delta b_0 \to ((\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to b) \ldots))).$$

We can obtain by continuing

$\Delta b'_1 \to (\Delta b'_2 \to \ldots (\Delta b'_m \to (a \to b)) \ldots) \leq \Delta b'_1 \to (\Delta b'_2 \to \ldots (\Delta b'_m \to (\Delta b_0 \to (\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to b) \ldots))) \ldots)$.

Then,

$$\Delta b'_1 \to (\Delta b'_2 \to \ldots (\Delta b'_m \to (\Delta b_0 \to (\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to b) \ldots))) \ldots) = \mathbf{1}.$$

And so $b \in M$. Finally, we will prove that $\Delta a \in M$ whenever $a \in M$. Assume that $a \in M$, then

$(\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b \to a) \ldots)) = \mathbf{1}$ for some $b_1, b_2, \ldots, b_n \in X$.

By (EΔ1), Lemma 2(b), Lemma 2(d), and the order properties of "→",

$$\begin{aligned} \mathbf{1} = \Delta\mathbf{1} &= \Delta(\Delta b_1 \to (\Delta b_2 \to \cdots (\Delta b_n \to a) \ldots)) \\ &\leq (\Delta\Delta b_1 \to (\Delta\Delta b_2 \to \cdots (\Delta\Delta b_n \to \Delta a) \ldots)) \\ &= (\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b \to \Delta a) \ldots)). \end{aligned}$$

Hence, $\Delta a \in M$. Therefore, M is a filter of $\mathcal{E}_\Delta$. Let $F \in \mathcal{F}(\mathcal{E}_\Delta), X \subseteq F$ and $a \in M$, then

$(\Delta b_1 \to (\Delta b_2 \to \ldots (\Delta b_n \to a) \ldots)) = \mathbf{1}$, for some $b_i \in X$ and $n \in \omega$.

Since $\mathbf{1}, \Delta b_1, \Delta b_2, \ldots, \Delta b_n \in F$, we imply $a \in F$. Thus, $M \subseteq F$. Therefore, M is the smallest filter of $\mathcal{E}_\Delta$ containing X. i.e. $M = \langle X \rangle$. ■

Theorem 3. Let F be a filter of a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra $\mathcal{E}_\Delta$. Then

$$\langle F, a \rangle = \{b \in E : \Delta a \to b \in F\}$$

Proof. Let $b \in \langle F, a \rangle$, then by Theorem 2 and Lemma 1(i) for some $f_1, f_2, \ldots, f_n \in F, n, k_1, k_2 \in \omega$

$\Delta f_1 \to (\Delta f_2 \to \ldots (\Delta f_n \to (\Delta a \to^{k_1} \tilde{b}^{k_2}) \ldots) = \mathbf{1}$.

Since F is a filter and $\mathbf{1} \in F$, then $\Delta a \to^{k_1} \tilde{b}^{k_2} \in F$. Hence, by Lemma 1(i) and Lemma 2(i) we get,

$\Delta a \to^{k_1} \tilde{b}^{k_2} \leq (\Delta a \otimes \ldots \otimes \Delta a) \to \tilde{b}^{k_3} = \Delta a \to \tilde{b}^{k_3} \in F$

for some $k_3 \in \omega$. Since F is a filter, then by Lemma 2(b), (d) and (g) and Lemma 3(a), we obtain

$\Delta(\Delta a \to \tilde{b}^{k_3}) \leq \Delta\Delta a \to \Delta\tilde{b}^{k_3} = \Delta a \to \Delta b \leq \Delta a \to b \in F$

Thus, $b \in \{b \in E : \Delta f \to (\Delta a \to b) = \mathbf{1}$ for some $f \in F\}$.

Conversely, since $\langle F, a \rangle$ is a filter, and $a \in \langle F, a \rangle$, then $\Delta a \in \langle F, a \rangle$. If $\Delta a \to b \in F$, then $\Delta a \to b \in \langle F, a \rangle$, and hence, $b \in \langle F, a \rangle$. ■

By the following theorem, we determine filters generated by join of two elements.

Theorem 4. Let F be a filter of a $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra $\mathcal{E}_\Delta$, and $a, b \in E$. Then

$$a \vee b \in F \text{ implies } \langle F, a \rangle \cap \langle F, b \rangle = F;$$

Proof. It is clear that $F \subseteq \langle F, a \rangle \cap \langle F, b \rangle$. Let $a \vee b \in F$, then by Definition 4 and Lemma 2(c), $\Delta(a \vee b) = \Delta a \vee \Delta b \in F$. Now let $c \in \langle F, a \rangle \cap \langle F, b \rangle$, then by Theorem 3, we get $\Delta a \to c \in F$ and $\Delta b \to c \in F$ for some $f \in F$. Hence, by Lemma 4(b), we have $(\Delta a \to c) \otimes (\Delta b \to c) \in F$. By this, Lemma 1(k) and Lemma 3(a), we have

$(\Delta a \to c) \otimes (\Delta b \to c) \leq (\Delta a \vee \Delta b) \to c \in F$.

Therefore, $c \in F$. Thus, $\langle F, a \rangle \cap \langle F, b \rangle \subseteq F$. ■

We extend to $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra the following result, proved by El-Zekey in [8]. The proof is completely the same as El-Zekey's.

Proposition 2. Let F be a filter of a prelinear $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra $\mathcal{E}_\Delta$. Then F is prime iff E/F is a chain, i.e., is linearly (totally) ordered by $\leq_F$.

Theorem 5. Let $\mathcal{E}_\Delta$ be a prelinear $\ell\mathrm{EQ}^{\mathrm{s}}_\Delta$-algebra and let $a \in E, a \neq \mathbf{1}$. Then, there is a prime filter F on $\mathcal{E}_\Delta$ not containing a.

Proof. There are filters not containing a, e.g. $F_0 = \{\mathbf{1}\}$. We shall show that if F is any filter not containing a and $x, y \in E$ such that $(x \to y) \notin F$ and $(y \to x) \notin F$, then there is a filter $F' \supseteq F$ not containing a but containing either $(x \to y) \in F$ or $(y \to x) \in F$. Note that the least filter F' containing F as a subset and $u \in E$ as an element is $F' = \{v \in E : \Delta u \to v \in F\}$. Indeed, F' is obviously a filter by Theorem 3 equivalently $F' = \langle F, u \rangle$.

Thus, assume $(x \to y) \notin F, (y \to x) \notin F$ and let F_1, F_2 be the smallest filters containing F as a subset and $(x \to y), (y \to x)$ respectively as an element. We claim that $a \notin F_1$ or $a \notin F_2$. Assume the contrary; then,

$\Delta(x \to y) \to a \in F$ and $\Delta(y \to x) \to a \in F$.

Hence, by Lemma 4(b), we have

$(\Delta(x \to y) \to a) \otimes (\Delta(y \to x) \to a) \in F$.

By this, Lemma 1(k) and Lemma 3(a), we have

$$\begin{aligned} (\Delta(x \to y) \to a) \otimes (\Delta(y \to x) \to a) &\leq (\Delta(x \to y) \vee \Delta(y \to x)) \to a \\ &= \mathbf{1} \to a \in F. \end{aligned}$$

Thus, $a \in F$ (since $\mathbf{1} \in F$) a contradiction. Hence $a \notin F_1$ or $a \notin F_2$.

Now, if $\mathcal{E}_\Delta$ is countable (which will be our case in the proof of completeness), then we may arrange all pairs (x, y) from E^2 into a sequence $\{(x_n, y_n) | n \text{ natural}\}$, put $F_0 = \{\mathbf{1}\}$ and having constructed F_n such that $p \notin F_n$ we take $F_{n+1} \supseteq F_n$ such that $p \notin F$ according to our construction; if possible we take F_{n+1} such that $(x_n \to y_n) \in F_{n+1}$, if not, we take that with $(y_n \to x_n) \in F_{n+1}$. Our desired prime filter is the union

$$\bigcup_n F_n$$

If $\mathcal{E}_\Delta$ is uncountable, then one has to use the axiom of choice and work similarly with a transfinite sequence of filters. ■

Theorem 6. (Representation theorem). Let $\mathcal{E}_\Delta$ be a prelinear $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra. Then, each $\mathcal{E}_\Delta$ is subdirectly embeddable into a product of linearly ordered $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebras; i.e., $\mathcal{E}_\Delta$ is representable.

Proof. Let $\boldsymbol{\mathcal{P}}$ be the set of all prime filters of $\mathcal{E}_\Delta$. For $F \in \boldsymbol{\mathcal{P}}$. Thus, by Theorem 1, the natural homomorphism $h: \mathcal{E}_\Delta \to \prod_{F\in\boldsymbol{\mathcal{P}}} \mathcal{E}_\Delta/\approx_F$ defined by $h(a) = \langle [a]_F \rangle_{F\in\boldsymbol{\mathcal{P}}}$ is a subdirect embedding of $\mathcal{E}_\Delta$ into a direct product of $\{\mathcal{E}_\Delta/\approx_F : F \in \boldsymbol{\mathcal{P}}\}$. It remains to show that it is one-one. If $a, b \in F$ and $a \neq b$ then $a \nleq b$ or $b \nleq a$. Without loss of generality, then $(a \to b) \neq$ **1** in E. By Theorem 5, let F be a prime filter on E not containing $(a \to b)$; then in $\mathcal{E}_\Delta/F$, $[a]_F \nleq [b]_F$, hence $[a]_F \neq [b]_F$ and therefore $h(a) \neq h(b)$. Using Proposition 2 and Theorem 2, $\mathcal{E}_\Delta/\approx_F$ is linearly ordered $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebra for each $F \in \boldsymbol{\mathcal{P}}$, which completes the proofs. ■

4. Conclusions

In this paper, we introduced and studied a class of separated (not necessarily good) lattice EQ-algebras that may be represented as subalgebras of products of linearly ordered ones. Such algebras are called representable. Namely, we enriched separated lattice EQ-algebras with a unary operation (the so called Baaz delta), fulfilling some additional assumptions. The resulting algebras are called $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebras. One of the main results of this paper is to characterize the class of representable $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebras. We showed that prelinearity alone characterizes the representable class of $\ell\mathrm{EQ}_\Delta^{\mathrm{s}}$-algebras. We also supplied a number of useful results, leading to this characterization.

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